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# Fundamental solutions of homogeneous elliptic differential operators ☆

Brice Camus

*Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstr. 150, D-44780 Bochum, Germany*

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**Abstract**

We compute fundamental solutions of homogeneous elliptic differential operators, with constant coefficients, on $\mathbb{R}^n$ by mean of analytic continuation of distributions. The result obtained is valid in any dimension, for any degree and can be extended to pseudodifferential operators of the same type.


*Keywords:* Fundamental solutions; PDE

Let be $P = P(D_x)$ a pseudodifferential operator, with constant coefficients, obtained by mathematical quantization of the function $p$, i.e.:

$$Pf(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\langle \xi, x\rangle} p(\xi) \hat{f}(\xi)\, \mathrm{d}\xi. \tag{1}$$

Here, and in the following, the notation:

$$\hat{f}(\xi) = \int\limits_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle y, \xi\rangle} f(y)\, \mathrm{d}y,$$

designs the Fourier transform of $f$. We note $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space and $\mathcal{S}'(\mathbb{R}^n)$ the distributions on $\mathcal{S}(\mathbb{R}^n)$. A fundamental solution for $P$ is a distribution $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^n)$ such that $P\mathfrak{S} = \delta$. Fundamental solutions play a major role in the theory of PDE. For a large overview on this

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*E-mail address:* brice.camus@univ-reims.fr (B. Camus).

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subject, and applications, we refer to [1,2]. Apart in some trivial cases, there is few explicit characterizations of fundamental solutions and most results concern the existence. The case of order 3 homogeneous operators, in dimension 3, was treated in [3]. Always for $n = 3$, the case of certain elliptic homogeneous operators of degree 4 was also solved in [4]. We are here interested in the case of a definite homogeneous polynomial $p_k$ on $\mathbb{R}^n$, i.e. $p_k(\xi) = 0 \Leftrightarrow \xi = 0$ and:

$$p_k(\lambda\xi) = |\lambda|^k p_k(\xi).$$

Note that $k$ has to be even but we do not impose the spherical symmetry of $p_k$. Strictly speaking, it is not necessary to assume that $k \in \mathbb{N}$ and we can consider elliptic operators with conical singularities at the origin. The main motivation is that such an operator can generalize the Laplacian (positivity and ellipticity) but these operators also play a role in optical physics. To state the main result, we introduce the spherical average of $g \in \mathcal{S}(\mathbb{R}^n)$ w.r.t. the symbol $p$:

$$A(g)(r) = r^{n-1} \int\limits_{\mathbb{S}^{n-1}} g(r\theta) p_k(\theta)^{-1} \, \mathrm{d}\theta. \tag{2}$$

Here $p_k(\theta)$ designs the restriction of $p_k$ on the sphere, which defines a strictly positive function. Clearly, we have $A(g) \in \mathcal{S}(\mathbb{R}_+)$ and we can extend $A(g)$ by 0 for $r < 0$ to obtain a $L^2$ function on the line. The main result is:

**Theorem 1.** *If $k < n$ a fundamental solution $\mathfrak{G} \in \mathcal{S}'(\mathbb{R}^n)$ for $P$ is given by*:

$$\langle \mathfrak{G}, f \rangle = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}_+ \times \mathbb{S}^{n-1}} p_k(\theta)^{-1} \hat{f}(r\theta) r^{n-1-k} \, \mathrm{d}r \, \mathrm{d}\theta.$$

*But, when $k \geqslant n$, we have*:

$$\langle \mathfrak{G}, f \rangle = -C_{k,n} \frac{\partial^{k-1} A(\hat{f})}{\partial r^{k-1}}(0) + D_{k,n} \int\limits_{\mathbb{R}_+} \log(u) \frac{\partial^k A(\hat{f})}{\partial r^k}(u) \, \mathrm{d}u,$$

*where $C_{k,n}$ and $D_{k,n}$ are universal constants given by Eqs.* (8), (9).

The reader can observe the analogy with the Laplacian, see e.g. [1]. In particular, one has to distinguish the case of an integrable (respectively non-integrable) singularity for $p_k(\xi)^{-1}$ for the $n$-dimensional Lebesgue measure.

**Proof of Theorem 1.** Since $p_k \geqslant 0$, we define a family $\mathfrak{p}(z)$ of distributions:

$$\forall f \in \mathcal{S}(\mathbb{R}^n) \colon \langle \mathfrak{p}(z), f \rangle = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} p_k(\xi)^z \hat{f}(\xi) \, \mathrm{d}\xi.$$

The r.h.s. is holomorphic for $\Re(z) > -1/k$ and, by continuity, we obtain:

$$\lim_{z \to 0} \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} p_k(\xi)^z \hat{f}(\xi) \, \mathrm{d}\xi = f(0) = \langle \delta, f \rangle. \tag{3}$$

The Laurent development of $\mathfrak{p}$ in $z = -1$ can be written:

$$\mathfrak{p}(z-1) = \sum_{j=-1}^{-d} z^j \mu_j + \mu_0 + \sum_{j=1}^{\infty} \mu_j z^j. \tag{4}$$

This point is justified in Lemma 2 below. But, according to Eq. (3), we have:

$$\lim_{z\to 0}\bigl\langle \mathfrak{p}(z-1), P(D)f\bigr\rangle = \langle \delta, f\rangle.$$

It is then easy to check that $\mu_0$ is a fundamental solution for $P$. Also, note that Eqs. (3), (4) provide the set of non-trivial relations:

$$P(D)\mu_j = 0, \quad \forall j < 0, \tag{5}$$

in the sense of distributions of $\mathcal{S}'(\mathbb{R}^n)$. The existence of such non-zero $\mu_j$, for $j < 0$, implies the non-uniqueness of fundamental solutions in $\mathcal{S}'(\mathbb{R}^n)$.

**Lemma 2.** *The distributions $\mathfrak{p}(z-1)$ are meromorphic on $\mathbb{C}$ with poles located at rational points $z_{j,k} = -\frac{j}{k}$, $j \in \mathbb{N}$.*

**Proof.** Let $g = \hat{f}$. Using standard polar coordinates we obtain:

$$\bigl\langle \mathfrak{p}(z-1), f\bigr\rangle = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}_+\times\mathbb{S}^{n-1}} \bigl(r^k p_k(\theta)\bigr)^{z-1} g(r\theta) r^{n-1}\,\mathrm{d}r\,\mathrm{d}\theta.$$

Here $p_k(\theta)$ is the restriction of $p_k$ to $\mathbb{S}^{n-1}$. Next, if we define:

$$y = (y_1, \ldots, y_n) = \bigl(r p_k(\theta)^{\frac{1}{k}}, \theta\bigr),$$

we obtain a very elementary formulation:

$$\bigl\langle \mathfrak{p}(z-1), f\bigr\rangle = \frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}_+} y_1^{k(z-1)} G(y_1)\,\mathrm{d}y_1.$$

This new amplitude $G$ is obtained by pullback and integration:

$$G(y_1) = \int y^*\bigl(g(r,\theta) r^{n-1}|Jy|\bigr)\,\mathrm{d}y_2\ldots\mathrm{d}y_n. \tag{6}$$

We have $G \in \mathcal{S}(\mathbb{R}^+)$ and $G(y_1) = \mathcal{O}(y_1^{n-1})$ in $y_1 = 0$. Starting from:

$$\frac{\partial^k}{\partial y_1^k} y_1^{kz} = \prod_{j=0}^{k-1}(kz-j) y_1^{k(z-1)},$$

after integrations by parts, we accordingly obtain that:

$$\bigl\langle \mathfrak{p}(z-1), f\bigr\rangle = \frac{1}{(2\pi)^n}\prod_{j=0}^{k-1}\frac{1}{(kz-j)}\int\limits_{\mathbb{R}_+} y_1^{kz}\partial_{y_1}^k G(y_1)\,\mathrm{d}y_1. \tag{7}$$

The integral of the r.h.s. is holomorphic in the strip $\Re(z) > -\frac{1}{k}$. Finally, we can iterate the previous construction to any order to obtain the result. $\square$

Since $z = 0$ is a simple pole, the constant term of the Laurent series is:

$$\frac{1}{(2\pi)^n k}\lim_{z\to 0}\partial_z\left(\prod_{j=1}^{k-1}\frac{1}{(kz-j)}\int\limits_{\mathbb{R}_+} y_1^{kz}\partial_{y_1}^k G(y_1)\,\mathrm{d}y_1\right).$$

For the calculations, we distinguish integrable and non-integrable singularities.

*Case of* $k < n$. The derivative of the rational function provides:

$$C_{k,n} = \frac{1}{(2\pi)^n k}\left(\partial_z \prod_{j=1}^{k-1} \frac{1}{(kz-j)}\right)\Bigg|_{z=0} = \frac{(-1)^{k+1}}{(2\pi)^n}\left(\frac{\gamma+\psi(k)}{\Gamma(k)}\right), \tag{8}$$

where $\gamma$ is Euler's constant and $\psi$ the poly-gamma function of order 0, i.e.:

$$\gamma = \lim_{m\to\infty} \sum_{j=1}^{m} \frac{1}{k} - \log(m),$$

$$\psi(z) = \partial_z\big(\log(\Gamma(z)\big) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Accordingly, since $G$ vanishes up to the order $n-1$ at the origin, we have:

$$C_{k,n} \int\limits_{\mathbb{R}_+} \partial_{y_1}^k G(y_1)\,\mathrm{d}y_1 = -C_{k,n}\partial_{y_1}^{k-1} G(0) = 0.$$

On the other side, by derivation of the integral, we find the term:

$$\frac{(-1)^{k-1}}{(k-1)!} \int\limits_{\mathbb{R}_+} \log(y_1)\partial_{y_1}^k G(y_1)\,\mathrm{d}y_1.$$

Since $k < n$, we can integrate by parts to finally obtain:

$$\int\limits_{\mathbb{R}_+} G(y_1)\frac{\mathrm{d}y_1}{y_1^k} = \int\limits_{\mathbb{R}_+\times\mathbb{S}^{n-1}} p_k(\theta)^{-1} g(r\theta) r^{n-1-k}\,\mathrm{d}r\,\mathrm{d}\theta.$$

Here, we have replaced the expression for our amplitude, via inversion of our diffeomorphism. This proves the first statement of Theorem 1. □

*Case of* $k \geqslant n$. The term attached to the derivative of the rational function is non-zero and provides:

$$-C_{k,n}\partial_{y_1}^{k-1} G(0) + \frac{1}{(2\pi)^n}\frac{(-1)^{k-1}}{(k-1)!} \int\limits_{\mathbb{R}_+} \log(y_1)\partial_{y_1}^k G(y_1)\,\mathrm{d}y_1.$$

We compute first the derivative of $G$. Contrary to the case $k < n$ we cannot take the limit directly but we will reach the result with the Schwartz kernel technic. By Fourier inversion formula we have:

$$\partial_{y_1}^{k-1} G(0) = \frac{1}{2\pi} \int \mathrm{e}^{-\mathrm{i}\xi u}(\mathrm{i}\xi)^{k-1} G(u)\,\mathrm{d}u\,\mathrm{d}\xi.$$

We can extend the integral w.r.t. $u$ on the whole line by extending $G$ by zero for $u < 0$. Going back to initial coordinates provides:

$$\partial_{y_1}^{k-1} G(0) = \frac{1}{2\pi} \int \mathrm{e}^{-\mathrm{i}\xi r}(\mathrm{i}\xi)^{k-1} r^{n-1} \int\limits_{\mathbb{S}^{n-1}} g(r\theta) p_k(\theta)^{-1}\,\mathrm{d}\theta\,\mathrm{d}r\,\mathrm{d}\xi.$$

The r.h.s. is exactly the derivative of order $k-1$ of $A(g)$ defined in Eq. (2). By the same technic, the logarithmic integral gives the non-local contribution:

$$\frac{1}{(k-1)!}\int_{\mathbb{R}_+}\log(u)\frac{\partial^k A(g)}{\partial r^k}(u)\,\mathrm{d}u.$$

In particular the second universal constant of Theorem 1 is:

$$D_{k,n}=\frac{1}{(2\pi)^n}\frac{(-1)^{k-1}}{(k-1)!}. \tag{9}$$

This proves the second assertion of Theorem 1. □

The main result holds also holds for elliptic pseudo-differential operators with homogeneous symbol of degree $\alpha \in ]0,\infty[$. Using $k = E(\alpha)+1$ integrations by parts in Eq. (7), where $E(x)$ is the integer part of $x$, the construction remains the same and all formulas can be analytically continued w.r.t. the degree $\alpha > 0$. This continuation is trivial since the functions of Eqs. (8), (9), defining respectively $C_{n,k}$ and $D_{n,k}$, are analytic w.r.t. the variable $k > 0$.

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